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Replica symmetry breaking in the random replicant model

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Abstract. We study the statistical mechanics of a model describing the co-evolution of species interacting in a random way. We find that at high competition, replica symmetry is broken. We solve the model in the approximation of one-step replica symmetry breaking and we compare our findings with accurate numerical simulations.

1. Introduction

Replicant models are used to study the co-evolution of sets of interacting species able to reproduce themselves: they have a huge number of applications in biological problems [1–3]. Furthermore, the processes that lead to the selection of a certain number of species through the interactions have been applied to computer optimization programs [4].

Until now, many problems (e.g. how to find the number of surviving species, to estimate the number of equilibrium configurations, or to study their stability) have been widely treated for several types of deterministic interactions [2, 3]. In this paper we study a non-deterministic evolution: we consider a system of replicants which evolve with random interactions.

The model introduced by Diederich and Opper [5] is defined as follows. Given N species, let x_i/N be the concentration of the ith family in the system. The real variables $\{x_i \in \mathbb{R}, i = 1, \ldots, N\}$ are then subject to the constraints

$$\sum_{i=1}^{N} x_i = N \qquad x_i \geqslant 0 \ \forall i = 1, \ldots, N. \tag{1.1}$$

The interactions between different species are described through a fitness function $F_J[\boldsymbol{x}]$ that must be maximized at equilibrium. Typically, F_J is chosen as a quadratic function of the concentrations; this is equivalent to considering only pair interactions between the species.

Taking into account the reproduction of the species, the evolution equations are

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = x_i \left(F_{J,i} - \langle F_J \rangle \right) \qquad i = 1, \ldots, N \tag{1.2}$$

where the derivative

$$F_{J,i} := \frac{\partial}{\partial x_i} F_J(x_1, \ldots, x_N) \tag{1.3}$$

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measures how much the configuration $\{x_1, \dots, x_N\}$ enhances the ith species, and

$$\langle F_J \rangle := \frac{1}{N} \sum_{i=1}^{N} F_{J,i} \tag{1.4}$$

denotes the average growth of the system that must be subtracted from $F_{J,i}$ in order to keep the normalization (1.1).

In the random replicant model, the fitness funtional F_J introduces random interactions between each couple of species:

$$F_J[x] = -\mathcal{H}_J[x] := -\sum_{i<j=1}^{N} J_{ij} x_i x_j - a \sum_{i=1}^{N} x_i^2 \tag{1.5}$$

where the parameters $\{J_{ij}\}$ are chosen at random from the Gaussian probability distribution

$$P(J_{ij}) = \sqrt{\frac{N}{\pi J^2}} \exp\left(-\frac{N J_{ij}^2}{J^2}\right) \tag{1.6}$$

as in the Sherrington–Kirkpatrick (SK) model of spin glasses [6, 7]. Note that the introduction of a symmetric function in which $J_{ij} = J_{ji}\ \forall i \neq j$ is a very strong approximation from the biological point of view, since in many cases the interaction between the species is by no means symmetric.

The control parameter a has the aim of limiting the growth and the supremacy of one single species: for large values of a, the growth of all the species is strongly limited by the factor ax_i^2; in that case, the random interactions become negligible and the equilibrium configuration is

$$x_i^{\mathrm{eq}} \simeq 1 \qquad \forall i = 1, \dots, N \qquad (a \gg J) \tag{1.7}$$

i.e. almost independent of the interactions between the species. Instead, for small values of a, the pair interactions play a central role and a few species prevail among the others. Analytically, this model differs from the SK spin glass in that we impose the constraint (1.1): the spins are then allowed to take any real value, but the total magnetization is fixed.

In section 2 we show how it is possible to solve the random replicant model within the replica formalism. In sections 3 and 4 we analyse the replica symmetric solution and its stability, and in section 5 we perform the first step of the hierarchical replica symmetry breaking. The biological applications of the results are found in the limit $T \to 0^+$ because the fitness function F_J is, a minus sign apart, the low temperature limit of the free energy.

The study of the stability of the replica symmetric solution will show that, at zero temperature, the replicant model exhibits a phase transition to a glassy phase when a crosses a certain value a_c. The replica symmetry breaking which occurs in the glassy phase ($a < a_c$) implies the breakdown of the ergodicity of the system: when a becomes small, the evolution of the system depends strongly on the initial conditions and, in general, we will not be able to make any precise prediction on the equilibrium state of the system.

From the biological point of view, the glassy phase is the unstable phase: in the high a phase a single equilibrium state exists and the system is able to recover its equilibrium configuration after any external change of the concentrations of its elements; in contrast, in the glassy phase, the same perturbation can drastically change the final configuration of the system if it is led to a different ergodic region of the phase space. Here, however, we study only the properties of the statics associated to Hamiltonian (1.5) and we do not consider the dynamics of a system leading to this equilibrium distribution.

2. The random replicant model: analytical solution

Now we derive the expression for the quenched free-energy density of the random replicant model. In this and the next section we closely follow [5]. The evolution of the system is ruled by the Hamiltonian (1.5). Averaging over all the possible choices of the $\{J_{ij}\}$, the quenched free energy of the system is given by

$$-\beta N f = \int \prod_{i<j} \mathrm{d}J_{ij}\, P(J_{ij}) \ln \sum_{[x]} \exp(-\beta \mathcal{H}_J[x]) . \tag{2.1}$$

To compute (2.1) we use the replica method [7–12], introducing a set of Lagrange multipliers $\{\lambda_\alpha, \alpha = 1, \ldots, n\}$ which ensure the normalization condition (1.1) in each of the n replicas. With standard calculations [13], we arrive at the following expression for f:

$$-\beta f = \lim_{n\to 0^+} \max_{Q,\lambda} \left[-\frac{1}{n}\sum_{\alpha\gamma} Q^2_{\alpha\gamma} + \frac{1}{n}\sum_\alpha \lambda_\alpha + \frac{1}{n}\ln \mathrm{Tr}_n \exp \mathrm{L}(Q, \lambda, x)\right] \tag{2.2}$$

where

$$\mathrm{L}(Q, \lambda, x) := -\beta a \sum_\alpha x^2_\alpha - \sum_\alpha \lambda_\alpha x_\alpha + \beta J \sum_{\alpha\gamma} Q_{\alpha\gamma} x_\alpha x_\gamma \tag{2.3}$$

Q and λ are, respectively, an $n\times n$ matrix and an n-dimensional vector; $\{x_\alpha, \alpha = 1, \ldots, n\}$ is a new set of real positive variables, and Tr_n denotes the integral over all possible values of the x_α's.

From (2.2) and (2.3) we find that the stationary equations for f are

$$\begin{cases} Q_{\alpha\gamma} = \dfrac{\beta J}{2}\dfrac{\mathrm{Tr}_n\, x_\alpha x_\gamma \exp \mathrm{L}(Q,\lambda,x)}{\mathrm{Tr}_n \exp \mathrm{L}(Q,\lambda,x)} & \forall \alpha,\gamma = 1,\ldots,n \\ 1 = \dfrac{\mathrm{Tr}_n\, x_\alpha \exp \mathrm{L}(Q,\lambda,x)}{\mathrm{Tr}_n \exp \mathrm{L}(Q,\lambda,x)} & \forall \alpha = 1,\ldots,n. \end{cases} \tag{2.4}$$

The remaining sections are devoted to the study of the solutions of these stationary conditions.

3. Replica symmetric solution

Both the free-energy density and the above stationary equations are invariant under the action of the group S_n of permutations between the n replicas. This implies that at least one of the solutions of (2.4) is invariant under S_n, and so the first ansatz that is to be tried is certainly the symmetric one, which is given by

$$Q_{\alpha\gamma} = q\delta_{\alpha\gamma} + t \qquad \text{and} \qquad \lambda_\alpha = \lambda. \tag{3.1}$$

Introducing (3.1) into (2.2), and denoting the resulting free-energy density by f_{RS} we have, after the manipulations described in [13],

$$-\beta f_{\mathrm{RS}} = \max_{q,\tilde{t},\tilde{\lambda}} \left[q^2 + 2\beta J q\tilde{t} - \beta J\tilde{\lambda} - \ln\left(\int_0^{+\infty} \mathrm{d}x\, \exp \mathcal{L}_{\mathrm{RS}}(q,\tilde{t},\tilde{\lambda},x,z)\right)\right] \tag{3.2}$$

where

$$\mathcal{L}_{\rm RS}(q,\tilde{t},\tilde{\lambda},x,z) := -\beta J\left[(\tilde{a}-q)x^2 - \left(2z\sqrt{\tilde{t}}-\tilde{\lambda}\right)x\right] \tag{3.3}$$

$$\tilde{t} := t/(\beta J) \qquad \tilde{\lambda} := \lambda/(\beta J) \qquad \tilde{a} := a/J$$

and we have introduced the notation

$$\overline{G(z)} := \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} \mathrm{d}z\, \mathrm{e}^{-z^2} G(z). \tag{3.4}$$

In a similar way, the stationary equations become

$$\begin{cases} \overline{\langle x\rangle_z} = 1 \\ \overline{\langle x\rangle_z^2} = 2\tilde{t} \\ \overline{\langle x^2\rangle_z - \langle x\rangle_z^2} = \dfrac{2q}{\beta J} \end{cases} \tag{3.5}$$

where

$$\langle G(x)\rangle_z := \frac{\int_0^{+\infty}\mathrm{d}x\, G(x)\exp \mathcal{L}_{\rm RS}(q,\tilde{t},\tilde{\lambda},x,z)}{\int_0^{+\infty}\mathrm{d}x\,\exp\mathcal{L}_{\rm RS}(q,\tilde{t},\tilde{\lambda},x,z)}. \tag{3.6}$$

The low temperature limit of the symmetric solution, first studied by Diederich and Opper [5], is particularly interesting because it allows us to prove analytically the existence of a second-order transition to a glassy phase, as we will show in the next section. Introducing the parameter

$$\tilde{z} := \frac{\tilde{\lambda}}{2\sqrt{\tilde{t}}} \tag{3.7}$$

the stationary equations (3.5) become

$$\begin{cases} 4q(\tilde{a}-q) = \dfrac{1}{\sqrt{\pi}}\displaystyle\int_{\tilde{z}}^{+\infty}\mathrm{d}z\,\mathrm{e}^{-z^2} \\ 4\tilde{t}(2q-\tilde{a}) = \tilde{\lambda} \\ \dfrac{\sqrt{\tilde{t}}\mathrm{e}^{-\tilde{z}^2}}{\sqrt{\pi}} - 2(\tilde{a}-q) = \dfrac{\tilde{\lambda}}{\sqrt{\pi}}\displaystyle\int_{\tilde{z}}^{+\infty}\mathrm{d}z\,\mathrm{e}^{-z^2} \end{cases} \tag{3.8}$$

leading to $f_{\rm RS} = 2J\tilde{t}(\tilde{a}-2q)$. Figure 1 shows how q, $\tilde{t}$, $\tilde{\lambda}$, and $f_{\rm RS}$ behave as functions of $\tilde{a}$ in this limit. We also give the approximate expressions of these parameters in two particularly interesting cases: the 'classical' regime ($\tilde{a} \gg 1$), and the neighbourhood of the critical point $\tilde{a}_{\rm c} = 1/\sqrt{2}$.

In the former region, the equilibrium configurations become trivial, with $x_i^{\rm eq} \simeq 1$, $\forall i$. The replica symmetric solution, which we will prove to be stable in this region, predicts

$$\begin{aligned} q &= \tfrac{1}{2}\left(\tilde{a}-\sqrt{\tilde{a}^2-1}\right) + \mathcal{O}\left(\mathrm{e}^{-\tilde{a}^2}\right) \\ \tilde{t} &= \frac{1}{4}\left(1+\frac{\tilde{a}}{\sqrt{\tilde{a}^2-1}}\right) + \mathcal{O}\left(\mathrm{e}^{-\tilde{a}^2}\right) \\ \tilde{\lambda} &= -\tilde{a}-\sqrt{\tilde{a}^2-1} + \mathcal{O}\left(\mathrm{e}^{-\tilde{a}^2}\right) \end{aligned} \tag{3.9}$$

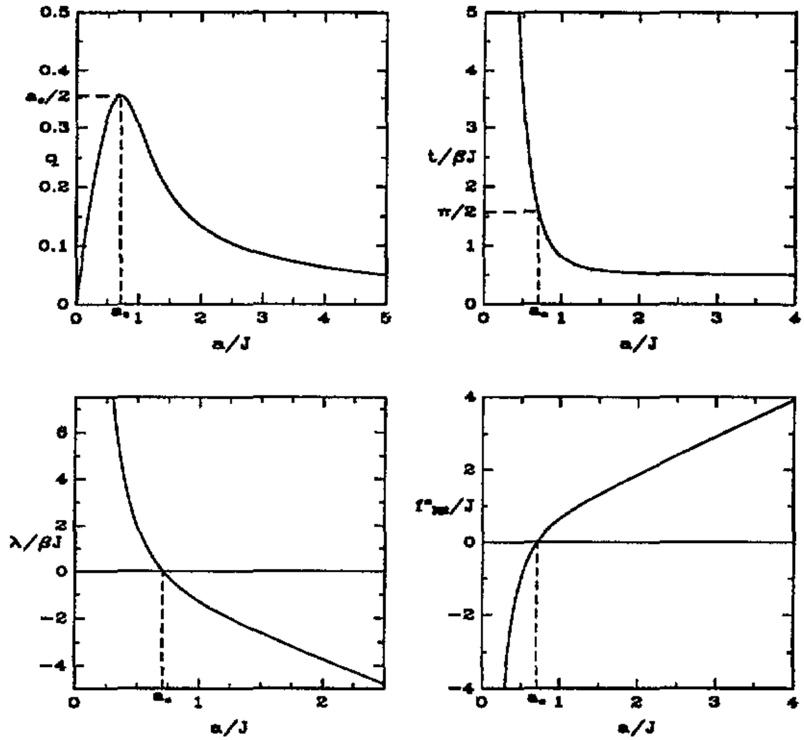

Figure 1. Numerical solutions of the replica symmetric equations in the low temperature limit.

and the free-energy density becomes

$$f_{\mathrm{RS}}^{o} = \tfrac{1}{2}\left(\tilde{a} + \sqrt{\tilde{a}^2 - 1}\right) + \mathcal{O}\left(\mathrm{e}^{-\tilde{a}^2}\right). \tag{3.10}$$

Instead, the latter is the transition point to the glassy phase, as we will show below. In its neighbourhood we have

$$\begin{aligned}
q &= \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{2}(\pi - 2)(\tilde{a} - \tilde{a}_{\mathrm{c}})^2 + \mathrm{O}\left((\tilde{a} - \tilde{a}_{\mathrm{c}})^2\right) \\
\tilde{t} &= \frac{\pi}{2} - \sqrt{2}\pi(\pi - 2)(\tilde{a} - \tilde{a}_{\mathrm{c}}) + \pi^2(3\pi - 8)(\tilde{a} - \tilde{a}_{\mathrm{c}})^2 + \mathrm{O}\left((\tilde{a} - \tilde{a}_{\mathrm{c}})^2\right) \\
\tilde{\lambda} &= -2\pi(\tilde{a} - \tilde{a}_{\mathrm{c}}) + 2\sqrt{2}\pi(\pi - 2)(\tilde{a} - \tilde{a}_{\mathrm{c}})^2 + \mathrm{O}\left((\tilde{a} - \tilde{a}_{\mathrm{c}})^2\right) \\
f_{\mathrm{RS}}^{o} &= \pi(\tilde{a} - \tilde{a}_{\mathrm{c}}) - \sqrt{2}\pi(\pi - 2)(\tilde{a} - \tilde{a}_{\mathrm{c}})^2 + \mathrm{O}\left((\tilde{a} - \tilde{a}_{\mathrm{c}})^2\right).
\end{aligned} \tag{3.11}$$

Finally, figure 2 shows the numerical results that we obtained for the order parameters q and t by solving equations (3.5) for different finite values of β.

4. Instability of the symmetric solution

In the preceding section we have shown that equations (2.4) admit a symmetric solution, but we must also check the Hessian of the free energy to determine whether our solution is a

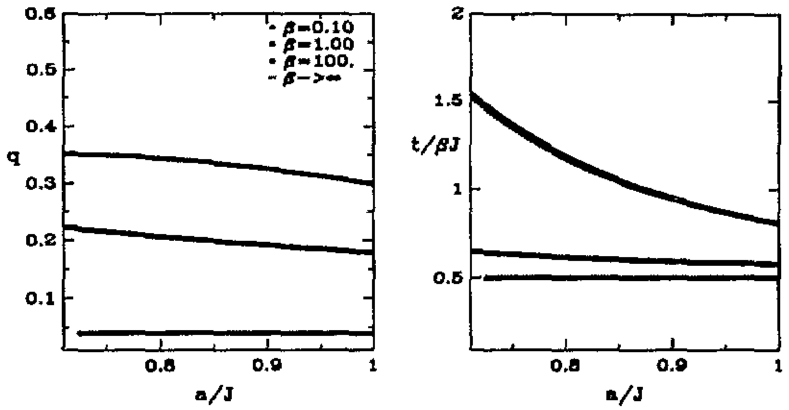

Figure 2. Numerical solutions of the replica symmetric equations at finite temperature.

minimum of f or just a saddle-point. To find the eigenvalues of the Hessian we generalize the calculus made by de Almeida and Thouless [9] for the SK model of spin glass: let

$$Q_{\alpha\gamma} = (q + \delta q_\alpha)\delta_{\alpha\gamma} + t + \delta t_{\alpha\gamma} \qquad \text{with} \begin{cases} \delta t_{\alpha\alpha} = 0 \\ \delta t_{\alpha\gamma} = \delta t_{\gamma\alpha} \end{cases} \qquad (4.1)$$

$$\lambda_\alpha = \lambda + \delta\lambda_\alpha .$$

If we denote the vector $(\boldsymbol{\delta\lambda}; \boldsymbol{\delta q}; \boldsymbol{\delta t})$ by $\boldsymbol{\delta\xi}$ and we substitute (4.1) in (2.2) we obtain, after some tedious calculations [13], that the second-order term in the expansion of f in terms of $\boldsymbol{\delta\xi}$ is given by $-\beta\delta_2 f = \frac{1}{2}\delta\xi^{\mathrm{T}} \cdot \mathcal{M}\delta\xi$, where $\mathcal{M}$ is a real symmetric matrix with the following fourteen different types of elements:

$$A := \mathcal{M}_{\delta\lambda_\alpha\delta\lambda_\alpha} = \left(\overline{\langle x^2\rangle_z} - \overline{\langle x\rangle_z}^2\right)$$

$$B := \mathcal{M}_{\delta\lambda_\alpha\delta\lambda_\gamma} = \left(\overline{\langle x\rangle_z^2} - \overline{\langle x\rangle_z}^2\right)$$

$$C := \mathcal{M}_{\delta\lambda_\alpha\delta q_\alpha} = -\beta J\left(\overline{\langle x^3\rangle_z} - \overline{\langle x^2\rangle_z\langle x\rangle_z}\right)$$

$$D := \mathcal{M}_{\delta\lambda_\alpha\delta q_\gamma} = -\beta J\left(\overline{\langle x^2\rangle_z\langle x\rangle_z} - \overline{\langle x^2\rangle_z}\,\overline{\langle x\rangle_z}\right)$$

$$E := \mathcal{M}_{\delta q_\alpha\delta q_\alpha} = -2 + \beta^2 J^2\left(\overline{\langle x^4\rangle_z} - \overline{\langle x^2\rangle_z}^2\right)$$

$$F := \mathcal{M}_{\delta q_\alpha\delta q_\gamma} = -\beta^2 J^2\left(\overline{\langle x^2\rangle_z^2} - \overline{\langle x^2\rangle_z}^2\right) \qquad (4.2)$$

$$G := \mathcal{M}_{\delta\lambda_\alpha\delta t_{\alpha\gamma}} = -\beta J\left(\overline{\langle x^2\rangle_z\langle x\rangle_z} - \overline{\langle x\rangle_z^2\langle x\rangle_z}\right)$$

$$H := \mathcal{M}_{\delta\lambda_\alpha\delta t_{\gamma\delta}} = -\beta J\left(\overline{\langle x\rangle_z^3} - \overline{\langle x\rangle_z^2\langle x\rangle_z}\right)$$

$$I := \mathcal{M}_{\delta q_\alpha\delta t_{\alpha\gamma}} = \beta^2 J^2\left(\overline{\langle x^3\rangle_z\langle x\rangle_z} - \overline{\langle x^2\rangle_z\langle x\rangle_z^2}\right)$$

$$J := \mathcal{M}_{\delta q_\alpha \delta t_{\gamma\delta}} = \beta^2 J^2 \left(\overline{\langle x^2\rangle_z \langle x\rangle_z^2} - \overline{\langle x^2\rangle_z \langle x\rangle_z^2}\right)$$

$$K := \mathcal{M}_{\delta t_{\alpha\gamma}\delta t_{\alpha\gamma}} = -2 + \beta^2 J^2 \left(\overline{\langle x^2\rangle_z^2} - \overline{\langle x\rangle_z^2}^2\right)$$

$$K' := \mathcal{M}_{\delta t_{\alpha\gamma}\delta t_{\gamma\alpha}} = \beta^2 J^2 \left(\overline{\langle x^2\rangle_z^2} - \overline{\langle x\rangle_z^2}^2\right) = K + 2$$

$$L := \mathcal{M}_{\delta t_{\alpha\gamma}\delta t_{\gamma\delta}} = \beta^2 J^2 \left(\overline{\langle x^2\rangle_z \langle x\rangle_z^2} - \overline{\langle x\rangle_z^2}^2\right)$$

$$M := \mathcal{M}_{\delta t_{\alpha\gamma}\delta t_{\delta\eta}} = \beta^2 J^2 \left(\overline{\langle x\rangle_z^4} - \overline{\langle x\rangle_z^2}^2\right).$$

Furthermore, $\mathcal{M}$ has three different types of eigenvectors.

(i) Symmetric eigenvectors of the type

$$\delta\xi = (\ell, \dots, \ell; \rho, \dots, \rho; \tau, \dots, \tau) \tag{4.3}$$

(ii) 1-asymmetry eigenvectors, with

$$\begin{aligned}
\delta\lambda_\alpha &= \begin{cases} \ell_1 & \text{if } \alpha = \tilde{\alpha} \\ \ell_0 & \text{otherwise} \end{cases} \\
\delta q_\alpha &= \begin{cases} \rho_1 & \text{if } \alpha = \tilde{\alpha} \\ \rho_0 & \text{otherwise} \end{cases} \\
\delta t_{\alpha\gamma} &= \begin{cases} \tau_1 & \text{if } \alpha = \tilde{\alpha} \text{ or } \gamma = \tilde{\alpha} \\ \tau_0 & \text{otherwise} \end{cases}
\end{aligned} \tag{4.4}$$

(iii) 2-asymmetry eigenvectors, of the type

$$\begin{aligned}
\delta\lambda_\alpha &= \begin{cases} \ell_1 & \text{if } \alpha = \tilde{\alpha} \text{ or } \alpha = \tilde{\gamma} \\ \ell_0 & \text{otherwise} \end{cases} \\
\delta q_\alpha &= \begin{cases} \rho_1 & \text{if } \alpha = \tilde{\alpha} \text{ or } \alpha = \tilde{\gamma} \\ \rho_0 & \text{otherwise} \end{cases} \\
\delta t_{\alpha\gamma} &= \begin{cases} \tau_2 & \text{if } \alpha\gamma = \tilde{\alpha}\tilde{\gamma} \text{ or } \alpha\gamma = \tilde{\gamma}\tilde{\alpha} \\ \tau_0 & \text{if } \alpha \neq \tilde{\alpha}, \alpha \neq \tilde{\gamma}, \gamma \neq \tilde{\alpha} \text{ and } \gamma \neq \tilde{\gamma} \\ \tau_1 & \text{otherwise.} \end{cases}
\end{aligned} \tag{4.5}$$

The eigenvalues of $\mathcal{M}$ must be negative in order to ensure the stability of the symmetric ansatz. The biggest eigenvalues associated with the families above comes from the 2-asymmetry eigenvectors, and is given by [13]

$$\mu_{\rm cr} = \tfrac{1}{2}(K + K' - 2L + M) = -1 + \beta^2 \overline{\left(\langle x^2\rangle_z - \langle x\rangle_z^2\right)^2}. \tag{4.6}$$

In the low temperature limit $\mu_{\rm cr}$ can be easily computed, and is equal to

$$\mu_{\rm cr} = \frac{2q - \tilde{a}}{\tilde{a} - q}. \tag{4.7}$$

In particular, as figure 3 shows, $\mu_{\rm cr}$ becomes positive when $\tilde{a} < \tilde{a}_{\rm c}$:

$$\mu_{\rm cr} = \pi(\tilde{a} - \tilde{a}_{\rm c}) - \sqrt{2}\pi(\pi - 2)(\tilde{a} - \tilde{a}_{\rm c})^2 + O\left((\tilde{a} - \tilde{a}_{\rm c})^2\right). \tag{4.8}$$

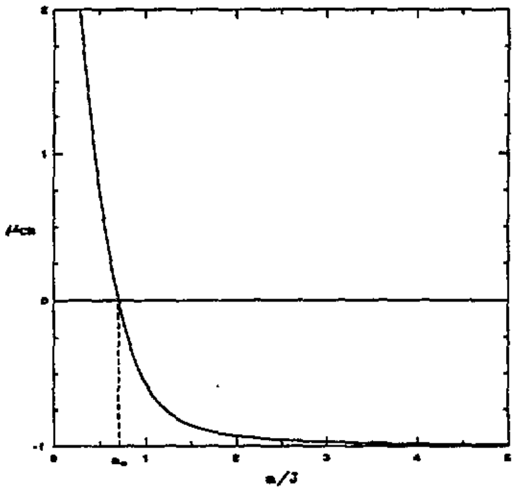

Figure 3. *Critical eigenvalue at zero temperature. The symmetric solution becomes unstable when it is positive.*

5. Replica symmetry breaking

Having proved the instability of the symmetric solution, we must now search for a more general ansatz to describe the system when $\tilde{a} < \tilde{a}_c$. To obtain it, we will follow the guidelines of the hierarchical ansatz of spin glasses [10–12]. In this paper we only study the first step of the replica symmetry breaking, testing order parameter matrices of the type

$$Q_{\alpha\gamma} = \begin{cases} t & \text{if Int}\left(\frac{\alpha}{\eta}\right) \neq \text{Int}\left(\frac{\gamma}{\eta}\right) \\ t + r & \text{if Int}\left(\frac{\alpha}{\eta}\right) = \text{Int}\left(\frac{\gamma}{\eta}\right) \text{ but } \alpha \neq \gamma \\ q + t + r & \text{if } \alpha = \gamma. \end{cases} \tag{5.1}$$

This ansatz can be improved by iterating the breaking scheme in all the blocks introduced in the first step, but we will see that even a single breaking drastically improves the symmetric predictions. We recall that, in the limit $n \to 0^+$, the hierarchical parametrization can be written in terms of an order parameter function $Q(x)$, defined in the interval $x \in [0, 1]$, which, at this point of symmetry breaking, is equal to

$$(x) = \begin{cases} t & \text{if } x \in [0, \eta) \\ t + r & \text{if } x \in (\eta, 1). \end{cases} \tag{5.2}$$

In (5.2) we have omitted the diagonal term containing q (corresponding to $Q(1)$), because it involves the term in the Hamiltonian that contains a and it can always be treated separately. The introduction of the breaking parameters η and r changes the free-energy density:

$$-\beta f_{\mathrm{H}} = \max_{q,t,r,\lambda,\eta} \Bigg[-(q+t+r)^2 - (\eta - 1)(t+r)^2 + \eta t^2 + \lambda + \frac{1}{\eta} \int_{-\infty}^{+\infty} \mathrm{d}z \, \frac{\mathrm{e}^{-z^2}}{\sqrt{\pi}} \ln \int_{-\infty}^{+\infty} \mathrm{d}z_r \, \frac{\mathrm{e}^{-z_r^2}}{\sqrt{\pi}} \left(\int_0^{+\infty} \mathrm{d}x \, \exp \mathcal{L}_{\mathrm{H}}(q,t,r,\lambda,x,z,z_r) \right)^{\eta} \Bigg] \tag{5.3}$$

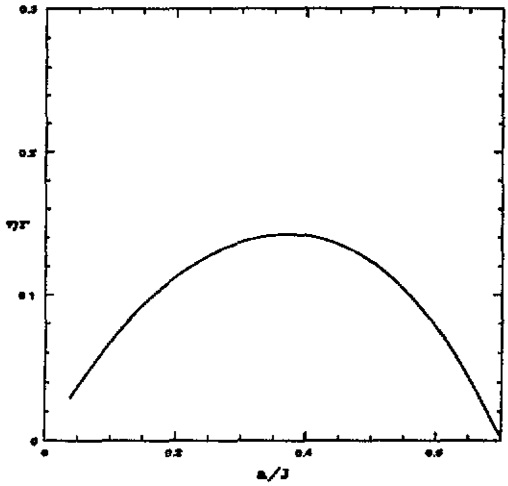

Figure 4. Numerical solutions obtained for the product of the breaking parameters η and r at zero temperature.

with

$$\mathcal{L}_{\mathrm{H}}(q,t,r,\lambda,x,z,z_r) := -\beta J(\tilde{a}-q)x^2 + \left(2z\sqrt{\beta J t} + 2z_r\sqrt{\beta J r} - \lambda\right)x. \tag{5.4}$$

The stationary equations related to f_{H} become

$$\begin{aligned}
1 &= \overline{\left[\langle x\rangle_{(z,z_r)}\right]_z} \\
t &= \frac{\beta J}{2}\overline{\left[\langle x\rangle_{(z,z_r)}\right]_z^2} \\
r &= \frac{\beta J}{2}\overline{\left(\left[\langle x\rangle^2_{(z,z_r)}\right]_z - \left[\langle x\rangle_{(z,z_r)}\right]_z^2\right)} \\
q &= \frac{\beta J}{2}\overline{\left(\left[\langle x^2\rangle_{(z,z_r)}\right]_z - \left[\langle x\rangle^2_{(z,z_r)}\right]_z\right)} \\
\eta^2 r(r+2t) &= \overline{\left(\eta\left[\ln\int_0^{+\infty}\mathrm{d}x\,\mathcal{L}_{\mathrm{H}}\right]_z - \ln\int_{-\infty}^{+\infty}\mathrm{d}z_r\,\frac{\mathrm{e}^{-z_r^2}}{\sqrt{\pi}}\mathrm{P}(z,z_r)\right)}
\end{aligned} \tag{5.5}$$

where

$$\begin{aligned}
\mathrm{P}(z,z_r) &:= \left(\int_0^{+\infty}\mathrm{d}x\,\mathcal{L}_{\mathrm{H}}(q,t,r,\lambda,x,z,z_r)\right)^{\eta} \\
\langle\cdot\rangle_{(z,z_r)} &:= \frac{\int_0^{+\infty}\mathrm{d}x\cdot\mathcal{L}_{\mathrm{H}}(q,t,r,\lambda,x,z,z_r)}{\int_0^{+\infty}\mathrm{d}x\,\mathcal{L}_{\mathrm{H}}(q,t,r,\lambda,X,z,z_r)} \\
[\cdot]_z &:= \int_{-\infty}^{+\infty}\mathrm{d}z_r\,\frac{\mathrm{e}^{-z_r^2}}{\sqrt{\pi}}\cdot\mathrm{P}(z,z_r)\Big/\int_{-\infty}^{+\infty}\mathrm{d}z_r\,\frac{\mathrm{e}^{-z_r^2}}{\sqrt{\pi}}\mathrm{P}(z,z_r).
\end{aligned} \tag{5.6}$$

Solving equations (5.5) numerically in the low temperature limit, we find that the product ηr of the two breaking parameters remains finite in the $\beta\to+\infty$ limit, and that it becomes different from zero as soon as $\tilde{a} < \tilde{a}_{\mathrm{c}}$, as is shown in figure 4.

In figure 5 we show one of the results found in the numerical simulations that were performed on this model, and that we will describe in more detail elsewhere. The triangles represent the free-energy density obtained from the simulations at zero temperature, the full curve corresponds to the replica symmetric prediction, and the broken curve illustrates the broken symmetry results. Figure 5 clearly shows how the first step of the replica symmetry breaking improves the symmetric predictions, even if it fails when $\tilde{a}$ goes to zero.

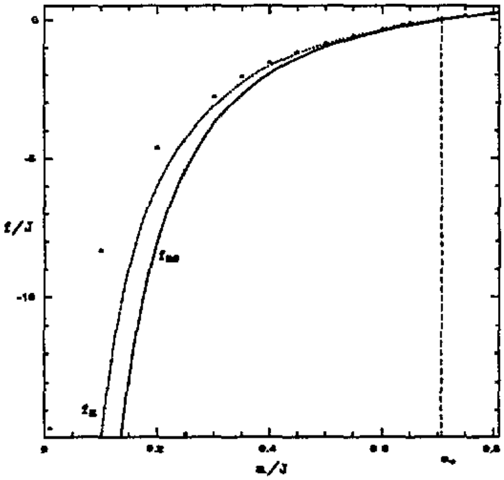

Figure 5. Improvement led by the one-step replica symmetry broken solution in the prediction for the free energy at zero temperature.

To conclude the study of the replica symmetry broken solution we will now show that, in the low temperature limit, η goes to zero as $\mathcal{O}(\beta^{-1})$, so that the breaking parameter r scales as t and λ do, i.e. that $r = \beta\tilde{r}$ with $\tilde{r}$ finite as T goes to zero. We will prove this result near the critical value of a where we have a better analytic control. To this end, we push our expansion of f to the third order in $\delta\lambda$, $b\delta q$, δt, obtaining

$$-\beta f = -\beta f_{\rm RS} + \lim_{n\to 0^+} \frac{f^{(2)} + f^{(3)}}{n}. \tag{5.7}$$

The second-order term $f^{(2)}$ was studied in the preceding section; considering the third-order term as a function of the order parameter function $Q(x)$, and neglecting higher-order terms in n, the stationary equation

$$\frac{\delta}{\delta Q(x)} f[q] = 0 \tag{5.8}$$

that must be verified $\forall x \in [0, 1]$, can be given in the form [13]

$$2\dot{Q}(x)\Bigg\{1 - \beta^2\left(\overline{\langle x^2\rangle_z} - \overline{\langle x\rangle_z^2}\right)^2 + 2\ell\beta^2\left(\overline{\langle x^3\rangle_z} - 3\overline{\langle x^2\rangle_z\langle x\rangle_z} + 2\overline{\langle x\rangle_z^3}\right)\left(\overline{\langle x^2\rangle_z} - \overline{\langle x\rangle_z^2}\right)$$
$$+ 2\rho\beta^3\left(\overline{\langle x^4\rangle_z} - 2\overline{\langle x^3\rangle_z\langle x\rangle_z} - \overline{\langle x^2\rangle_z^2} + 2\overline{\langle x\rangle_z^2\langle x^2\rangle_z}\right)\left(\overline{\langle x^2\rangle_z} - \overline{\langle x\rangle_z^2}\right)$$

$$
\begin{aligned}
&+2\beta^3 Q(x)\Bigg(2\left(\overline{\langle x^2\rangle_z}-\overline{\langle x\rangle_z^2}\right)^3 x-\left(\overline{\langle x^3\rangle_z}-\overline{\langle x^2\rangle_z\langle x\rangle_z}\right)\\
&\times\left(\overline{\langle x^3\rangle_z}-5\overline{\langle x^2\rangle_z\langle x\rangle_z}+4\overline{\langle x\rangle_z^3}\right)+4\overline{\langle x\rangle_z^2}\left(\overline{\langle x^2\rangle_z}-\overline{\langle x\rangle_z^2}\right)^2\Bigg)\\
&+4\beta^3\int_0^1 \mathrm{d}x'\,Q(x')\left(\overline{\langle x^3\rangle_z}-3\overline{\langle x^2\rangle_z\langle x\rangle_z}+2\overline{\langle x\rangle_z^3}\right)\left(\overline{\langle x^2\rangle_z}-\overline{\langle x\rangle_z^2}\right)\\
&+4\beta^3\int_x^1 \mathrm{d}x'\,Q(x')\left(\overline{\langle x^2\rangle_z}-\overline{\langle x\rangle_z^2}\right)^3\Bigg\}=0
\end{aligned}
\tag{5.9}
$$

where

$$
\dot{Q}(x) := \frac{\mathrm{d}Q}{\mathrm{d}x}. \tag{5.10}
$$

One can take the derivative of this expression with respect to x to obtain a necessary condition for the equilibrium:

$$
\begin{cases}
\dot{Q}(x)=0\\
\text{or}\\
-4\beta^3\left[2x\left(\overline{\langle x^2\rangle_z}-\overline{\langle x\rangle_z^2}\right)^3-\left(\overline{\langle x^3\rangle_z}-3\overline{\langle x^2\rangle_z\langle x\rangle_z}+2\overline{\langle x\rangle_z^3}\right)^2\right]=0.
\end{cases}
\tag{5.11}
$$

This is exactly what we were looking for: the order parameter function $Q(x)$ must be a constant in all $x \in [0, 1]$, except for

$$
x=\eta:=\frac{\left(\overline{\langle x^3\rangle_z}-3\overline{\langle x^2\rangle_z\langle x\rangle_z}+2\overline{\langle x\rangle_z^3}\right)^2}{2\left(\overline{\langle x^2\rangle_z}-\overline{\langle x\rangle_z^2}\right)^3}
\tag{5.12}
$$

where a jump can happen. Note that in Ising spin glasses with two-spin interactions no solution of this type can be found, while a similar phenomenon happens in the Potts model with p components, when $p > 4$.

The integrals in (5.12) can be computed in the low temperature limit, leading to [13]

$$
\eta=\frac{1}{\beta^3J^3(\tilde{a}-q)^3}\sqrt{\frac{\tilde{a}-q}{\tilde{\imath}\pi}}\frac{\mathrm{e}^{-\tilde{z}^2}c_{\mathrm{N}}}{\beta J}\Bigg/\frac{4q(\tilde{a}-q)}{4\beta^3J^3(\tilde{a}-q)^3}=\frac{\mathrm{e}^{-\tilde{z}^2}c_{\mathrm{N}}}{\beta J q\sqrt{\tilde{\imath}\pi(\tilde{a}-q)}}
\tag{5.13}
$$

where $\tilde{z}$ is defined in (3.7), and the numerical constant c_{N} can be easily evaluated, and is equal to 0.016 7066.... Equation (5.13) shows that the scaling behaviour of η is precisely $\eta = \mathcal{O}(\beta^{-1})$ when $\beta \to +\infty$.

6. Conclusions

We have shown that in the replicant model replica symmetry is broken. The predictions based on one-step replica symmetry breaking are in better agreement with the numerical data than those coming from exact symmetry, but in this case the one-step replica symmetry breaking gives only a partial improvement of the theoretical predictions in the limit of very small a. It would be rather interesting to obtain the results from full replica symmetry breaking in this region. This task should not be impossible using the techniques of [13].

The replica symmetric phase, which at zero temperature is stable only for $\tilde{a} \geqslant \tilde{a}_c = \sqrt{2}/2$, is characterized as usual by the presence of only one equilibrium configuration. When the evolution of the system is studied in this phase, the final configuration does not depend either on the starting point or on the details of the evolution, because no free-energy barriers break the ergodicity of the system, and there is no degeneracy of the minima of the energy.

In contrast, in the disordered phase, ergodicity is broken, a huge number of metastable equilibrium configurations appear, and the evolution of the system is determined by the singularities of the free-energy surface in the phase space. Furthermore, accurate numerical simulations can give more detailed information about this phase; even when many metastable states appear, the state with minimum free energy always has a wider attraction basin than the upper states. Furthermore, when the number of minima of the energy becomes very great (i.e. for very small a), the surviving probability of one single species depend on the starting point of the evolution. The most surprising feature is that this dependence is extremely low until we reach values of $\tilde{a}$ of the order of $10^{-1}\tilde{a}_c$. Finally, the ground-state energy gave us a quantitative measure of how the replica symmetry breaking improves our theoretical predictions: a single step breaking almost halves the errors of the symmetric results (see figure 5).

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